A stability approach for reaction-diffusion problems

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***Abstract*—In this paper we investigate the N-stability notion in an abstract Banach space setting. The main result is that we verify the N-stability of the implicit difference method for the periodic initial-value reaction-diffusion problem.**

I. Introduction

Many phenomena in nature such as physical, biological or chemical processes can be described by mathematical models. In chemistry one of the most investigated problem is the reaction-diffusion problem. Reaction-diffusion is a process in which two or more chemicals diffuse over a surface and react with one another to produce stable patterns. However (in general), the solution of such models cannot be given in a closed form, therefore we construct numerical models in order to approximate the exact solution. Generally, consistency in itself is not enough for convergence. To guarantee this property we introduce the notion of stability.

The stability property is verified for periodic initial-value reaction-diffusion problem in case of globally Lispchitz continuous forcing function f in several works, e.g. in Ascher [1] and Thomas [8]. Regarding the stability proof, these books use the fact that we know the eigenvalues of the standard matrix replacement of the second derivative operator with periodic boundary conditions. Further techniques are introduced e.g., discrete time Fourier transform ([8]).

In the recent years we dealt with the investigation of the numerical solution of nonlinear operator equations in an abstract (Banach space) setting. This work has been summarized in [3], [4]. In this paper our primary aim is to describe the discretizated version of the above mentioned problem and verify the N-stability in an abstract setting. Thanks to this method we can offer different approach for verifying the stability of nonlinear equations.

This paper is organized as follows. In Section II we give a short mathematical formulation of general description of the mathematical and numerical models. Moreover, we define the basic numerical notions (convergence, consistency). In Section III we introduce the notion of N-stability for reaction-diffusion problems based on the work of Sans-Serna ([6], [7]). In Section III-A we treat the case without the forcing function with the implicit finite difference method. Furthermore we show that this method is N-stable. In Section III-B we verify the N-stability with the above mentioned method in the case of including the forcing term (the so-called IMEX-method). In Section IV we make some remarks and we also summarize our results.

II. Mathematical background

We consider the problem

$$F(u) = 0\,, \tag{1}$$

where $F : \mathcal{D} \to \mathcal{Y}$ is a (nonlinear) operator, $\mathcal{D} \subset \mathcal{X}$, $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces. In the theory of numerical analysis it is usually *assumed* that there exists a unique solution, which will be denoted by $\bar{u}$.

Definition 1. *Problem* (1) *can be identified as a triplet* $\mathscr{P} = (\mathcal{X}, \mathcal{Y}, F)$. *We will refer to it as* problem $\mathscr{P}$.

Definition 2. *The sequence* $\mathscr{N} = (\mathcal{X}_n, \mathcal{Y}_n, F_n)_{n\in\mathbb{N}}$ *is called numerical method if it generates a sequence of problems*

$$F_n(u_n) = 0\,, \quad n = 1, 2, \dots\,, \tag{2}$$

where

- $\mathcal{X}_n, \mathcal{Y}_n$ *are normed spaces,*
- $\mathcal{D}_n \subset \mathcal{X}_n$ *and* $F_n : \mathcal{D}_n \to \mathcal{Y}_n$.

If there exists a unique solution of (2)*, it will be denoted by* $\bar{u}_n$.

Remark 3. *In the sequel we always assume the existence of* $\bar{u}$ *and* $\bar{u}_n$.

Definition 4. *We say that the sequence* $\mathscr{D} = (\varphi_n, \psi_n, \Phi_n)_{n\in\mathbb{N}}$ *is a* discretization *of problem* $\mathscr{P}$ *by numerical method* $\mathscr{N}$ *, if*

- *the* φ_n*-s (respectively* ψ_n*-s) are restriction operators from* $\mathcal{X}$ *into* $\mathcal{X}_n$ *(respectively from* $\mathcal{Y}$ *into* $\mathcal{Y}_n$*),*
- $\Phi_n : \{F : \mathcal{D} \to \mathcal{Y} \mid \mathcal{D} \subset \mathcal{X}\} \to \{F_n : \mathcal{D}_n \to \mathcal{Y}_n \mid \mathcal{D}_n \subset \mathcal{X}_n\}$.

In our paper we always do the following assumption for the general discretization $\mathscr{D}$.

Assumption 5.

(a) *The discretization* $\mathscr{D}$ *possesses the property* $\psi_n(0) = 0$.
(b) *The discretization* $\mathscr{D}$ *generates the numerical method* $\mathscr{N}$ *with the property* $\dim \mathcal{X}_n = \dim \mathcal{Y}_n < \infty$.

In sense of Definitions 1, 2 and 4 we can illustrate the general scheme, showed in Figure 1.

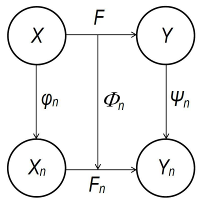

Fig. 1. The general scheme of numerical methods.

(c) *F_n is continuous on some ball $B_R(\varphi_n(\bar{u}))$.*

Remark 6. *Obviously, when ψ_n are linear operators, then the Assumption 5 (a) is automatically satisfied.*

Definition 7. *The element $e_n = \varphi_n(\bar{u}) - \bar{u}_n \in \mathcal{X}_n$ is called global discretization error.*

Definition 8. *The discretization $\mathscr{D}$ applied to the problem $\mathscr{P}$ is called* convergent *if*

$$\lim \|e_n\|_{\mathcal{X}_n} = 0$$

holds.

Definition 9. *The element $l_n(v) = F_n(\varphi_n(v)) - \psi_n(F(v)) \in \mathcal{Y}_n$ is called local discretization error at the element $v \in \mathcal{D}$. The local discretization error on the solution, i.e., $l_n(\bar{u}) = F_n(\varphi_n(\bar{u})) - \psi_n(F(\bar{u})) = F_n(\varphi_n(\bar{u}))$ is called local discretization error.*

Definition 10. *The discretization $\mathscr{D}$ applied to problem $\mathscr{P}$ is called consistent on the element $v \in D$ if*

- *$\varphi_n(v) \in \mathcal{D}_n$ holds from some index,*
- *for these elements the relation*

$$\lim \|l_n(v)\|_{\mathcal{Y}_n} = 0$$

holds.

III. N-STABILITY OF THE REACTION-DIFFUSION PROBLEM

In numerical analysis one of the most important task is to guarantee the convergence of the sequence of the numerical solutions. Generally, consistency in itself is not enough for convergence. To guarantee this property we introduce the notion of stability.

According to Definition 7, the convergence yields that the global discretization error e_n tends to zero. Having consistency, we have information about the local discretization error, only. Intuitively, this means the following requirement. When $l_n(\bar{u}) = F_n(\varphi_n(\bar{u})) - F(\bar{u}_n)$ is small, then $e_n = \varphi(\bar{u}) - \bar{u}_n$ should be small, too. Because $\bar{u}$ is unknown, therefore in first approach we require this property for any pairs in $\mathcal{D}_n$, i.e., for all $z_n,\ w_n \in \mathcal{D}_n$ we require the inequality

$$\|z_n - w_n\|_{\mathcal{X}_n} \le S\,\|F_n(z_n) - F_n(w_n)\|_{\mathcal{Y}_n}\,, \qquad (3)$$

where the so-called *stability constant* S is independent of the mesh-size parameter.

This idea leads to define the following nonlinear stability notion.

Definition 11. *The discretization $\mathscr{D}$ is called* N-stable *for the problem $\mathscr{P}$ if there exists positive stability constant S, such that for each $z_n, w_n \in \mathcal{D}_n$, the estimation* (3) *holds.*

Furthermore we will refer to this notion as the naive stability (N-stability).

Theorem 12. *We assume that*

- *there exists the solution of the problem* (1)-(2)*,*
- *the discretization $\mathscr{D}$ is consistent and N-stable at $\bar{u}$ with constant S on problem $\mathscr{P}$.*

Then the discretization $\mathscr{D}$ is convergent for the problem $\mathscr{P}$.

Proof. Due to the N-stability, we have the relation

$$\|e_n\|_{\mathcal{X}_n} = \|\varphi_n(\bar{u}) - \bar{u}_n\|_{\mathcal{X}_n} \le S\|F_n(\varphi_n(\bar{u})) - \underbrace{F_n(\bar{u}_n)}_{=0}\|_{\mathcal{Y}_n},$$

which leads to the estimation

$$\|e_n\|_{\mathcal{X}_n} \le S\,\|F_n(\varphi_n(\bar{u}))\|_{\mathcal{Y}_n} = S\,\|l_n(\bar{u})\|_{\mathcal{Y}_n}$$

and hence, for consistent methods this implies the convergence. □

A. Reaction-diffusion problem without forcing term

The N-stability is a natural stability notion for the nonlinear case. We have already shown that under the consistency the N-stability guaranties the convergence. Hence, we can use it to prove the convergence of several consistent method. Particularly, we can apply this approach to prove the convergence of the implicit finite difference scheme for the reaction-diffusion problem, too. Therefore, in the rest of the paper our aim is to verify the N-stability of the reaction-diffusion problem both with and without the forcing term.

Consider the periodic initial-value reaction-diffusion problem:

$$\partial_t u(t,x) = \partial_{xx} u(t,x), \quad x \in \mathbb{R},\ t \in [0,T], \qquad (4)$$

$$u(t,x) = u(t,x+1), \quad x \in \mathbb{R},\ t \in [0,T], \qquad (5)$$

$$u(0,x) = u^0(x), \quad x \in \mathbb{R}, \qquad (6)$$

where $T \in \mathbb{R}^+$. The conditions (5)-(6) are periodic boundary conditions and initial-value conditions, where u^0 is a given one-periodic function. It is easy to see that the continuous problem (4)-(6) can be rewritten in the form of (1). As we have mentioned, we assume the existence of unique, sufficiently smooth solution of the problem (4)-(6).

Remark 13. *Since the solution is periodic, it is sufficient to determine the solution in one period, only.*

To create the discretization $\mathscr{D}$ on the above mentioned problem we define both the spatial and time grids, as follows. The spatial grid points are

$$\{x_j = jh,\ \text{where}\ j = 1,\dots,n,\ h = 1/n\ \text{and}\ n \in \mathbb{N},\ n \ge 2\},$$

and the time levels are

$$\{t_k = k\delta, \text{ where } k=0,\dots,K \text{ and } \delta = T/K\}.$$

Applying the implicit difference method to reaction-diffusion problem (4)-(6), we get for $j=1,\dots,n$ and $k=0,\dots,K-1$,

$$\frac{u_j^{k+1}-u_j^k}{\delta} - \frac{u_{j-1}^{k+1}-2u_j^{k+1}+u_{j+1}^{k+1}}{h^2} = 0, \tag{7}$$

where using the periodic boundary conditions it is obvious that $u_0^{k+1} = u_n^{k+1}$ and $u_1^{k+1} = u_{n+1}^{k+1}$. The initial-value condition can be written as

$$u_j^0 - u^0(x_j) = 0, \quad j=1,\dots,n. \tag{8}$$

In the next step is that we rewrite (7)-(8) in the form of (2). To this aim we define the vector space of the grid functions $\mathcal{K}_n$, defined on the grid points $x_j : 1 \le j \le n$. If we consider u_j^k for the time level t_k for each k, then the denoted vector is $\mathbf{u}^k \in \mathcal{K}_n$. In Definition 4 we define $\varphi_n,\ \psi_n$ as the grid restriction operators. Hence, (7)-(8) can be written as

$$\frac{\mathbf{u}^{k+1}-\mathbf{u}^k}{\delta} - D_p^2\mathbf{u}^{k+1} = 0, \quad k=0,\dots,K-1, \tag{9}$$

$$\mathbf{u}^0 - \varphi_n(u^0) = 0, \tag{10}$$

where $\mathbf{u}^0 = (u^0(x_1),\dots,u^0(x_n)) \in \mathcal{K}_n$ and D_p^2 denotes the standard discretization matrix with periodic boundary conditions, i.e.,

$$D_p^2 = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

We choose in Definition 2: $\mathcal{X}_n = \mathcal{Y}_n = \underbrace{\mathcal{K}_n \times \ldots \times \mathcal{K}_n}_{K+1}$, hence $\mathbf{v}_n := (\mathbf{v}^0,\dots,\mathbf{v}^K) \in \mathcal{X}_n$. We introduce the following norms:

- in $\mathcal{K}_n$: $\|\mathbf{v}^k\|_{\mathcal{K}_n} = \max\limits_{1\le j\le n}\{v^k(x_j)\} = \|\mathbf{v}^k\|_\infty$,
- in $\mathcal{X}_n$: $\|\mathbf{v}_n\|_{\mathcal{X}_n} = \max\limits_{0\le k\le K}\|\mathbf{v}^k\|_{\mathcal{K}_n}$,
- in $\mathcal{Y}_n$: $\|\mathbf{v}_n\|_{\mathcal{Y}_n} = \|\mathbf{v}^0\|_{\mathcal{K}_n} + \sum\limits_{j=1}^{K}\delta\|\mathbf{v}^j\|_{\mathcal{K}_n}$.

We denote by $\eta_n = (\eta^0,\dots,\eta^K) \in \mathcal{Y}_n$ the image of $\mathbf{v}_n \in \mathcal{X}_n$. Due to this notation the mapping $F_n : \mathcal{X}_n \to \mathcal{Y}_n$ can be written as $F_n(\mathbf{v}_n) = \eta_n$. Particularly, for our discretization (9)-(10) it yields the following:

$$\frac{\mathbf{v}^{k+1}-\mathbf{v}^k}{\delta} - D_p^2\mathbf{v}^{k+1} = \eta^{k+1},\ k=0,\dots,K-1,$$

$$\mathbf{v}^0 = \eta^0.$$

The investigated method can be rewritten in the form

$$\mathbf{v}^{k+1} = Q^{-1}\mathbf{v}^k + \delta Q^{-1}\eta^{k+1}, \tag{11}$$

where $Q = I - \delta D_p^2$ is the subtransition matrix (which depend on h and δ). Introducing the notation $r = \delta/h^2$ we can write Q as

$$Q = \begin{pmatrix} 1{+}2r & -r & 0 & \cdots & 0 & 0 & -r \\ -r & 1{+}2r & -r & 0 & \cdots & 0 & 0 \\ 0 & -r & 1{+}2r & -r & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -r & 1{+}2r & -r & 0 \\ 0 & \cdots & \cdots & 0 & -r & 1{+}2r & -r \\ -r & 0 & 0 & \cdots & 0 & -r & 1{+}2r \end{pmatrix}.$$

Definition 14. *An $A \in \mathbb{R}^{n\times n}$ matrix is an M-matrix, if*

- *$a_{ij} \le 0$ for all $i \ne j$,*
- *there exists a vector $g \in \mathbb{R}^n,\ g > 0$ such that $Ag > 0$.*

Theorem 15. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix. Then*

$$\left\|A^{-1}\right\|_\infty \le \frac{\|g\|_\infty}{\min\limits_{1\le i\le n}(Ag)_i}. \tag{12}$$

Proof. It can be found in [2]. □

Since $r > 0$, thus Q is strictly diagonally dominant and $Q_{ij} \le 0$ for all $i \ne j$. Hence, Q is an M-matrix with the vector $g = (1,\dots,1)^T$. Due to (12), we have the estimate

$$\left\|Q^{-1}\right\|_\infty \le \frac{\|g\|_\infty}{\min\limits_{1\le i\le n}(Q^{-1}g)_i} = \frac{1}{1{+}2r{-}r{-}r} = 1. \tag{13}$$

Applying the recursion (11) and putting $\mathbf{v}^0 = \eta^0$ for arbitrary $k = 0,1,\dots,K$, we get

$$\mathbf{v}^k = (Q^{-1})^k\eta^0 + \sum_{j=1}^{k}\delta(Q^{-1})^j\eta^{k+1-j}.$$

Hence, according to the introduced norms, we obtain the estimation

$$\|\mathbf{v}_n\|_{\mathcal{X}_n} \le c\,\|\eta_n\|_{\mathcal{Y}_n}, \tag{14}$$

where

$$c = \max_{0\le k\le K}\left\{\max_{1\le j\le k}\left\{||(Q^{-1})^k||_\infty, ||(Q^{-1})^j||_\infty\right\}\right\}.$$

Using the relation (13), we get $||(Q^{-1})^k||_\infty \le ||Q^{-1}||_\infty^k \le 1$. Hence, obviously we have

$$\max_{0\le k\le K}\left\{\max_{1\le j\le k}\left\{||(Q^{-1})^k||_\infty, ||(Q^{-1})^j||_\infty\right\}\right\} = 1.$$

Since $F_n(\mathbf{v}_n) = \eta_n$, we can rewrite (14) as

$$\|\mathbf{v}_n\|_{\mathcal{X}_n} \le \|\eta_n\|_{\mathcal{Y}_n} = \|F_n(\mathbf{v}_n)\|_{\mathcal{Y}_n}. \tag{15}$$

For any elements $\mathbf{z}_n, \mathbf{w}_n \in \mathcal{X}_n$ we denote by ϱ_n and ξ_n their image, i.e., $F_n(\mathbf{z}_n) = \varrho_n$ and $F_n(\mathbf{w}_n) = \xi_n$. This results in the relations

$$\frac{\mathbf{z}^{k+1}-\mathbf{z}^k}{\delta} - D_p^2\mathbf{z}^{k+1} = \varrho^{k+1},\ k=0,\dots,K-1, \tag{16}$$

$$\mathbf{z}^0 - \varphi_n(u^0) = \varrho^0,$$

$$\frac{\mathbf{w}^{k+1}-\mathbf{w}^k}{\delta} - D_p^2\mathbf{w}^{k+1} = \xi^{k+1},\ k=0,\dots,K-1, \quad (17)$$

$$\mathbf{w}^0 - \varphi_n(u^0) = \xi^0.$$

Substracting (17) from (16), we gain for $k=0,\dots,K-1$

$$\mathbf{z}^{k+1}-\mathbf{w}^{k+1} = Q^{-1}(\mathbf{z}^k-\mathbf{w}^k) + \delta Q^{-1}(\varrho^{k+1}-\xi^{k+1}).$$

Using (15) by the notation $\mathbf{v}_n = \mathbf{z}_n - \mathbf{w}_n$, we gain

$$\|\mathbf{z}_n-\mathbf{w}_n\|_{\mathcal{X}_n} \le \|\varrho_n-\xi_n\|_{\mathcal{Y}_n} = \|F_n(\mathbf{z}_n)-F_n(\mathbf{w}_n)\|_{\mathcal{Y}_n}.$$

It is easy to see that the above estimation is in the form of (3) with $S=1$. Hence, the reaction-diffusion problem is N-stable. Then the following statement is true.

Theorem 16. *The implicit difference method is unconditionally convergent for the periodic initial-value reaction-diffusion problem* (4)-(6).

Proof. The implicit difference method is consistent and N-stable for this problem. Hence, due to Theorem 12, it is convergent, too. □

B. Reaction-diffusion problem with forcing term

Further we consider the following periodic initial-value reaction-diffusion problem:

$$\partial_t u(t,x) = \partial_{xx}u(t,x) + f(u), \quad x\in\mathbb{R},\ t\in[0,T], \quad (18)$$

$$u(t,x) = u(t,x+1), \quad x\in\mathbb{R},\ t\in[0,T], \quad (19)$$

$$u(0,x) = u^0(x), \quad x\in\mathbb{R}, \quad (20)$$

where $T\in\mathbb{R}^+$. In equation (18) we assume that $f:\mathbb{R}\to\mathbb{R}$ is a given globally Lipschitz continuous function. The conditions (19)-(20) are periodic boundary conditions and initial-value conditions, where u^0 is a given one-periodic function. It is easy to see that the continuous problem (18)-(20) can be rewritten in the form of (1). As we have mentioned, we assume the existence of unique, sufficiently smooth solution of the problem (18)-(20).

Remark 17. *Since the solution is periodic, it is sufficient to determine the solution in one period, only.*

Let us take the formerly introduced spatial and time grids and normes. We apply the IMEX-method based on the previous train of thought. For any elements $\mathbf{z}_n, \mathbf{w}_n \in \mathcal{X}_n$ we denote by ϱ_n and ξ_n their image, i.e., $F_n(\mathbf{z}_n)=\varrho_n$ and $F_n(\mathbf{w}_n)=\xi_n$. Then we consider the following two problems:

$$\frac{\mathbf{z}^{k+1}-\mathbf{z}^k}{\delta} - D_p^2\mathbf{z}^{k+1} - \mathbf{f}(\mathbf{z}^k) = \varrho^{k+1}, k=0,\dots,K-1, \quad (21)$$

$$\mathbf{z}^0 - \varphi_n(u^0) = \varrho^0,$$

$$\frac{\mathbf{w}^{k+1}-\mathbf{w}^k}{\delta} - D_p^2\mathbf{w}^{k+1} - \mathbf{f}(\mathbf{w}^k) = \xi^{k+1}, k=0,\dots,K-1, \quad (22)$$

$$\mathbf{w}^0 - \varphi_n(u^0) = \xi^0,$$

where $\mathbf{f}$ denotes the grid function defined on the grid points x_j, i.e., $[\mathbf{f}(\mathbf{z}^k)]_j = \varphi_n(f(x_j))$ for all $j=1,\dots,n$. Substracting (22) from (21), for $k=0,\dots,K-1$ we get

$$\begin{aligned}\mathbf{z}^{k+1}-\mathbf{w}^{k+1} &= Q^{-1}(\mathbf{z}^k-\mathbf{w}^k) + \delta Q^{-1}(\mathbf{f}(\mathbf{z}^k)-\mathbf{f}(\mathbf{w}^k)) \\ &\quad + \delta Q^{-1}(\varrho^{k+1}-\xi^{k+1}),\end{aligned} \quad (23)$$

where Q is the earlier defined subtransition matrix. Since f is globally Lipschitz continuous function, this implies the relation

$$\|\mathbf{f}(\mathbf{z}^k)-\mathbf{f}(\mathbf{w}^k)\|_{\mathcal{K}_n} \le L\|\mathbf{z}^k-\mathbf{w}^k\|_{\mathcal{K}_n}. \quad (24)$$

Due to (24) and applying the estimate (13), the recursion (23) shows that for $k=0,\dots,K$ we have the estimate

$$\begin{aligned}\left\|\mathbf{z}^k-\mathbf{w}^k\right\|_{\mathcal{K}_n} &\le (1+\delta L)\left\|\mathbf{z}^{k-1}-\mathbf{w}^{k-1}\right\|_{\mathcal{K}_n} \\ &\quad + \delta\left\|\varrho^k-\xi^k\right\|_{\mathcal{K}_n}.\end{aligned} \quad (25)$$

Applying recursion (25) and $\mathbf{z}^0-\mathbf{w}^0 = \varrho^0-\xi^0$, then for $k=0,\dots,K$ we get

$$\left\|\mathbf{z}^k-\mathbf{w}^k\right\|_{\mathcal{K}_n} \le (1+\delta L)^k \left\|\varrho_n-\xi_n\right\|_{\mathcal{Y}_n}.$$

Since $\delta K = T$, we get the estimation

$$\begin{aligned}\|\mathbf{z}_n-\mathbf{w}_n\|_{\mathcal{X}_n} &\le (1+\delta L)^K\|\varrho_n-\xi_n\|_{\mathcal{Y}_n} \le e^{L\delta K}\|\varrho_n-\xi_n\|_{\mathcal{Y}_n} \\ &= e^{LT}\|F_n(\mathbf{z}_n)-F_n(\mathbf{w}_n)\|_{\mathcal{Y}_n}.\end{aligned}$$

It is easy to see that the above estimation is in the form of (3) with $S=e^{LT}$. Hence, under the assumptions made, the reaction-diffusion problem is N-stable.

Theorem 18. *For the Lipschitzian forcing term f the IMEX-method is unconditionally convergent for the periodic initial-value reaction-diffusion problem* (18)-(20).

Proof. The IMEX-method is consistent and N-stable for this problem. Hence, due to Theorem 12, it is convergent, too. □

Remark 19. *Thanks to these results we can say if the given forcing term is in the form of $f(t,u)$ and it is a Lipschitz continuous function with respect to its second variable by the constant L, then we can similarly verify that the IMEX-method is convergent for the investigated problem.*

IV. Summary

In this paper we investigated the reaction-diffusion problem under certain conditions in an abstract setting. The base of our framework was the N-stablity notion. As we have shown in Section III, it is a useful notion regarding both theoretical results and applications. We verified that the implicit finite difference method is N-stable for the reaction-diffusion problem both with and without the forcing term.

The main advantage of this notion that we can offer another tool for verifying stability proprety for nonlinear equations. In the near future it is worth to deal with the N-stability of different physical, biological or chemical processes.

Acknowledgment

The work of the second author was supported by the Hungarian Research Grant OTKA K67819.

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