Stability concepts and their applications

Imre Fekete[a,b], István Faragó[a,b]

[a] *Department of Applied Analysis and Computational Mathematics, Eötvös Loránd University, H-1117 Budapest, Pázmány P. s. 1/C, Hungary*
[b] *MTA-ELTE Numerical Analysis and Large Networks Research Group, H-1117 Budapest, Pázmány P. s. 1/C, Hungary*

Abstract

The stability is one of the most basic requirement for the numerical model, which is mostly elaborated for the linear problems. In this paper we analyze the stability notions for the nonlinear problems. We show that, in case of consistency, both the N-stability and K-stability notions guarantee the convergence. Moreover, by using the N-stability we prove the convergence of the centralized Crank–Nicolson-method for the periodic initial-value transport equation. The K-stability is applied for the investigation of the forward Euler method and the θ-method for the Cauchy problem with lipschitzian right side.

Keywords: nonlinear stability, convergence, transport problem
2000 MSC: 65L20, 65M12, 65N12

1. Introduction and motivation

In order to solve the operator eqution, usually some numerical method is required, which means the construction of an adequate numerical model. One of the requirements for this model is stability, which seems to be one of the most challenging problems in numerical analysis. It is worth to emphasize that numerical stability is an intrinsic property of the numerical scheme and it is independent of the original continuous model. Commonly it is applied to the proof of convergence of the numerical method. (For this we need

☆Corresponding author Imre Fekete
Email addresses: `feipaat@cs.elte.hu` (Imre Fekete), `faragois@cs.elte.hu` (István Faragó)

Preprint submitted to Computers and Mathematics with Applications February 17, 2014

consistency, which establishes the link to the continuous problem.)

In the case of linear operators the first attempt was made by Kantorovich ([6]). The theory for this case is worked out and it is widely known (e.g., [8], [12]). However, the nonlinear theory is less elaborated. Stetter and Trenogin made the first attempts to define the notion of stability for nonlinear operators ([15], [17]). Later López-Marcos and Sanz-Serna began the systematic investigation of the basic numerical notions (consistency, stability and convergence) for nonlinear problems ([9], [14]). The abstract approach has stuck in. In the recent years we have made a similar approach to the investigation of the numerical solution of nonlinear operator equations in abstract settings. This work has been summarized in [1]. Thanks to these results and framework, we are able to use this approach to verify the stability of real-life problems. It is worth to mention that there are other approaches to treat the stability notion for nonlinear problems (see for more details in [11]).

When we model some real-life phenomenon with a mathematical model, it results in a - usually nonlinear - problem of the form

$$F(u) = 0\,, \tag{1.1}$$

where $F : \mathcal{D} \to \mathcal{Y}$ is a (nonlinear) operator, $\mathcal{D} \subset \mathcal{X}$, $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces. In the theory of numerical analysis it is usually *assumed* that there exists a unique solution, which will be denoted by $\bar{u}$. Problem (1.1) can be given as a triplet $\mathscr{P} = (\mathcal{X}, \mathcal{Y}, F)$. We will refer to it as the *problem* $\mathscr{P}$.

When we apply some numerical method, typically it generates a sequence of problems of the form

$$F_n(u_n) = 0\,, \quad n = 1, 2, \dots\,, \tag{1.2}$$

where $\mathcal{X}_n, \mathcal{Y}_n$ are normed spaces, $\mathcal{D}_n \subset \mathcal{X}_n$ and $F_n : \mathcal{D}_n \to \mathcal{Y}_n$. If there exists a unique solution of (1.2), it will be denoted by $\bar{u}_n$. We define the mappings $(\varphi_n)_{n\in\mathbb{N}}$ from $\mathcal{X}$ into $\mathcal{X}_n$ and $(\psi_n)_{n\in\mathbb{N}}$ from $\mathcal{Y}$ into $\mathcal{Y}_n$, respectively.

Definition 1.1. *The sequence of quintuples* $\mathscr{D} = (\mathcal{X}_n, \mathcal{Y}_n, F_n, \varphi_n, \psi_n)_{n\in\mathbb{N}}$ *is called a* discretization method.

In sense of this definition we can illustrate the general scheme, showed in Figure 1.1 (see, e.g. [5]).

For the convenience of the Reader, we formulate some basic definitions.

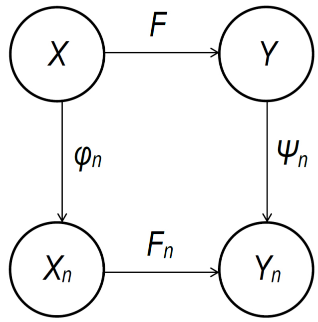

Figure 1.1: The general scheme of numerical methods.

Definition 1.2. *The element* $e_n = \varphi_n(\bar{u}) - \bar{u}_n \in \mathcal{X}_n$ *is called global discretization error. The element* $l_n(v) = F_n(\varphi_n(v)) - \psi_n(F(v)) \in \mathcal{Y}_n$ *is called local discretization error at the element* v*.*

Clearly the local discretization error on the solution is $l_n(\bar{u}) = F_n(\varphi_n(\bar{u}))$.

Definition 1.3. *We say that discretization* $\mathscr{D}$ *applied to the problem* $\mathscr{P}$ *is* convergent *if the relation*

$$\lim \|e_n\|_{\mathcal{X}_n} = 0$$

holds.

Definition 1.4. *The discretization* $\mathscr{D}$ *applied to problem* $\mathscr{P}$ *is called consistent on the element* $v \in D$ *if* $\varphi_n(v) \in \mathcal{D}_n$ *holds from some index and the relation*

$$\lim \|l_n(v)\|_{\mathcal{Y}_n} = 0$$

holds.

In numerical analysis one of the most important task is to guarantee the convergence of the sequence of the numerical solutions to the true solution $\bar{u}$. Generally, consistency in itself is not enough, therefore, to guarantee the convergence, we need certain additional condition. This is the notion of stability.

First of all we consider the sequence of linear problems, i.e., the problems

$$L_n(u_n) = 0\,, \quad n = 1, 2, \dots\,, \tag{1.3}$$

where for each n the operator L_n is linear and $L_n : \mathcal{D}_n \to \mathcal{Y}_n$. Naturally, we always assume the solvability of the problems (1.3), i.e., the existence of the operators $L_n^{-1} : \mathcal{Y}_n \to \mathcal{D}_n$. In this case, as it is known, the linear stability requires that $\|L_n^{-1}\|_{Lin(\mathcal{Y}_n,\mathcal{X}_n)} \leq S$ holds, where S is some positive constant.

Then the consistency and the stability together ensure the convergence. This result is well-known as the Lax (or sometimes Lax–Richtmyer–Kantorovich [8]) theorem. In numerical analysis it is also called as the "basic theory of numerical analysis".

2. Generalization of the stability notion

The linear stability notion implies some basic results. However, obtaining these consequences, we exploit the linearity of the operators L_n. In the rest of the paper our main aim is to study how to define the notion of stability in a suitable way for general (nonlinear) case.

2.1. First attempt: N-stability

The convergence yields that the global discretization error e_n tends to zero. Having consistency, we have information about the local discretization error only. Intuitively, this means that when $l_n(\bar{u}) = F_n(\varphi_n(\bar{u})) - F(\bar{u}_n)$ is small, then $e_n = \varphi(\bar{u}) - \bar{u}_n$ should be small, too. Because $\bar{u}$ is unknown, therefore in first approach we require this property for any pairs in $\mathcal{D}_n$. This demand implies the requirement

$$\|z_n - w_n\|_{\mathcal{X}_n} \leq S\,\|F_n(z_n) - F_n(w_n)\|_{\mathcal{Y}_n} \tag{2.4}$$

holds for arbitrary $z_n,w_n \in \mathcal{D}_n$ and the constant S is indepedent of the mesh size parameter.

This idea leads to make the first attempt to define the nonlinear stability notion.

Definition 2.1. *The discretization $\mathscr{D}$ is called* N-stable *on the problem $\mathscr{P}$ if there exists a positive S, called stability constant, such that for each $z_n, w_n \in \mathcal{D}_n$ the estimation (2.4) holds.*

Furthermore we will refer to this notion as the natural stability (N-stability). For the linear case the Definition 2.1 means the existence of a positive stability constant S, such that for each $s_n \in \mathcal{D}_n$

$$\|s_n\|_{\mathcal{X}_n} \leq S\,\|L_n(s_n)\|_{\mathcal{Y}_n} \tag{2.5}$$

holds. The bound (2.5) implies three basic properties:

i, For any problems (1.3), the relation (2.5) shows that $L_n(s_n) = 0$ implies that $s_n = 0$, i.e., L_n is injective and hence L_n^{-1} exists. If L_n is surjective, then the stability bound implies the existence and uniqueness of solutions of (1.3).

ii, Due to i, and (2.5), we have

$$\left\|L_n^{-1}(r_n)\right\|_{\mathcal{X}_n} \leq S \left\|r_n\right\|_{\mathcal{Y}_n}$$

for arbitrary $r_n \in \mathcal{Y}_n$. Therefore the uniform norm estimation

$$\left\|L_n^{-1}\right\|_{Lin(\mathcal{Y}_n,\mathcal{X}_n)} \leq S$$

holds.

iii, In view of (2.5), we obtain the "basic theory of numerical analysis":

$$\text{Consistency} + \text{Stability} \Rightarrow \text{Convergence.}$$

In fact, due to the linearity of L_n, by the choice $e_n = \varphi_n(\bar{u}) - \bar{u}_n$, we have

$$\left\|\varphi_n(\bar{u}) - \bar{u}_n\right\|_{\mathcal{X}_n} \leq S \left\|L_n(\varphi_n(\bar{u})) - L_n(\bar{u}_n)\right\|_{\mathcal{Y}_n},$$

which leads to the estimation

$$\left\|e_n\right\|_{\mathcal{X}_n} = \left\|\varphi_n(\bar{u}) - \bar{u}_n\right\|_{\mathcal{X}_n} \leq S \left\|L_n(\varphi_n(\bar{u}))\right\|_{\mathcal{Y}_n} = S \left\|l_n(\bar{u})\right\|_{\mathcal{Y}_n}.$$

Obviously, for consistent methods this implies the convergence.

The first two properties show that the linear stability notion is implied by the N-stability. On the other hand, the reverse implication is also true, since

$$\left\|s_n\right\|_{\mathcal{X}_n} = \left\|L_n^{-1}L_n(s_n)\right\|_{\mathcal{Y}_n} \leq \left\|L_n^{-1}\right\|_{Lin(\mathcal{Y}_n,\mathcal{X}_n)} \left\|L_n(s_n)\right\|_{\mathcal{Y}_n} \leq S \left\|L_n(s_n)\right\|_{\mathcal{Y}_n}.$$

Thanks to these results we can state that for linear problems the N-stability is equivalent to the linear stability notion. For the nonlinear case the following result is true.

Theorem 2.2. *We assume that*

- *there exists the solution of the problems* (1.1) *and* (1.2)*, respectively,*

- *the discretization $\mathscr{D}$ is consistent at $\bar{u}$ and N-stable with constant S on problem $\mathscr{P}$,*
- *for the mapping ψ_n the relation $\lim_{n\to\infty} \psi_n(0) = 0$ holds.*

Then the discretization $\mathscr{D}$ is convergent on problem $\mathscr{P}$ and the order of convergence is not less than the order of consistency.

Proof. Due to the N-stability, we have the relation

$$\|e_n\|_{\mathcal{X}_n} = \|\varphi_n(\bar{u}) - \bar{u}_n\|_{\mathcal{X}_n} \leq S\|F_n(\varphi_n(\bar{u})) - F_n(\bar{u}_n)\|_{\mathcal{Y}_n},$$

which leads to the estimation

$$\lim_{n\to\infty} \|e_n\|_{\mathcal{X}_n} \leq S \lim_{n\to\infty} \|F_n(\varphi_n(\bar{u})) - F_n(\bar{u}_n)\|_{\mathcal{Y}_n} \leq S \lim_{n\to\infty} \|l_n(\bar{u})\|_{\mathcal{Y}_n}.$$

Hence, for consistent methods the convergence is valid. □

Remark 2.3. *Formally, this statement can be written again as the "basic theory of numerical analysis":*

Consistency + N-Stability ⇒ Convergence.

There is a vital difference from the linear case, because Theorem 2.2 doesn't guarantee the existence of the numerical solution of equation (1.2).

As we have already seen, the N-stability is equivalent to the linear stability notion, and it satisfies the "basic theory of numerical analysis" for the nonlinear case. At the same time, it has a further advantageous property. Namely, it offers an alternative opportunity for verifying the stability of the numerical solution for time-dependent problems.

In the papers [3], [4] we investigated the N-stability property for periodic initial-value reaction-diffusion problem without the forcing term. Using the N-stability notion, we obtained the well-known stability results. It has been summarized in Table 2.1.

We also verified similar results for periodic initial-value reaction-diffusion problems, where the forcing term was a Lipschitz continuous function.

method		complexity	stability	convergence	
θ	name	explicit/implicit	$r = \delta/h^2$	time	space
0	forward Euler	explicit	$r \le 0.5$	1	1
1	backward Euler	implicit	$-$	1	1
0.5	Crank–Nicolson	implicit	$r \le 1$	1	2
θ	θ-method	explicit/implicit	$r \le 1/2(1-\theta)$	1	1 or 2

Table 2.1: The N-stability properties to the reaction-diffusion problem without the forcing term.

2.2. The N-stability of the transport problem

In the sequel, we apply the N-stability technique to verify the stability of hyperbolic equations, too, namely, to the periodic initial-value transport problem. We consider the problem

$$\begin{aligned} \partial_t u(t,x) + a\partial_x u(t,x) &= 0, & x \in \mathbb{R},\ t \in [0,T], \quad (2.6)\\ u(t,x) &= u(t,x+1), & x \in \mathbb{R},\ t \in [0,T] \quad (2.7)\\ u(0,x) &= u^0(x), & x \in \mathbb{R}, \quad (2.8) \end{aligned}$$

where $T \in \mathbb{R}^+$ and $a \in \mathbb{R}$ are fixed constants. The conditions (2.7)-(2.8) are periodic boundary conditions and initial-value conditions, where u^0 is a given one-periodic function. Periodic boundary condition appears in the stability investigation of the "good" Boussinesq equation in [10].

It is easy to see that the continuous problem (2.6)-(2.8) can be rewritten in the form (1.1). Let $u^0(x) \in C^1(\mathbb{R})$ be a given function, then the problem (2.6)-(2.8) has the unique solution $u(x,t) = u^0(x - at)$. Since the solution is periodic, it is sufficient to determine it on one period only. To create the discretization $\mathscr{D}$ on the above mentioned problem, we define both the spatial and time grids, as follows. The spatial grid points are

$$\{x_j = jh,\ \text{where } j = 1,\dots,n,\ h = 1/n \text{ and } n \in \mathbb{N},\ n \ge 2\},$$

and the time levels are

$$\{t_k = k\delta,\ \text{where } k = 0,\dots,K \text{ and } \delta = T/K\}.$$

Applying the centralized Crank–Nicolson-method to this transport problem, for $j=1,\dots,n$, and $k=0,\dots,K-1$, we gain the numerical scheme as follows

$$u_j^{k+1} + \frac{\delta a}{4h}\left(u_{j+1}^{k+1} - u_{j-1}^{k+1}\right) = u_j^k - \frac{\delta a}{4h}\left(u_{j+1}^{k} - u_{j-1}^{k}\right). \tag{2.9}$$

Using the periodic boundary conditions, we put $u_0^{k-1} = u_n^{k-1},\ u_1^{k-1} = u_{n+1}^{k-1}$ and $u_0^{k+1} = u_n^{k+1},\ u_1^{k+1} = u_{n+1}^{k+1}$. The discretization of the initial-value condition can be written as

$$u_j^0 - u^0(x_j) = 0, \quad j = 1,\dots,n. \tag{2.10}$$

In the next step we rewrite (2.9)-(2.10) in the form (1.2). To this aim, we define the vector space of the grid functions $\mathcal{K}_n$, defined at the grid points $x_j : 1 \le j \le n$. If we consider u_j^k for the time level t_k for each k, then the denoted vector is $\mathbf{u}^k \in \mathcal{K}_n$. We define the mappings φ_n and ψ_n as grid functions.

Introducing the notation $R = a\delta/h$ the equations (2.9)-(2.10) can be written as

$$\mathbf{u}^{k+1} + D_p\mathbf{u}^{k+1} = \mathbf{u}^k - D_p\mathbf{u}^k,\ k = 0,\dots,K-1, \tag{2.11}$$

$$\mathbf{u}^0 - \varphi_n(u^0) = 0, \tag{2.12}$$

where $\mathbf{u}^0 = (u^0(x_1),\dots,u^0(x_n)) \in \mathcal{K}_n$ and D_p denotes the standard discretization matrix with periodic boundary conditions, i.e.,

$$D_p = \begin{pmatrix} 0 & \frac{R}{4} & 0 & \cdots & 0 & 0 & -\frac{R}{4} \\ -\frac{R}{4} & 0 & \frac{R}{4} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{R}{4} & 0 & \frac{R}{4} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{R}{4} & 0 & \frac{R}{4} & 0 \\ 0 & \cdots & \cdots & 0 & -\frac{R}{4} & 0 & \frac{R}{4} \\ \frac{R}{4} & 0 & 0 & \cdots & 0 & -\frac{R}{4} & 0 \end{pmatrix}. \tag{2.13}$$

Introducing the notations $Q_1 = (I + D_p)$ and $Q_2 = (I - D_p)$, respectively, the discretization (2.11)-(2.12) yields the relation

$$Q_1 \mathbf{u}^{k+1} = Q_2 \mathbf{u}^k, \ k = 0, \ldots, K-1, \tag{2.14}$$
$$\mathbf{u}^0 = \varphi_n(u^0). \tag{2.15}$$

To prove the existence of the inverse of Q_1, we use the fact that D_p is a skew-symmetric matrix. Therefore its eigenvalues are on the imaginary axes, hence $Q_1 = (I + D_p)$ has no zero eigenvalue, and therefore it is regular. Then, we can rewrite (2.14)-(2.15) as

$$\begin{aligned} \mathbf{u}^{k+1} &= Q_1^{-1} Q_2 \mathbf{u}^k, \ k = 0, \ldots, K-1, \\ \mathbf{u}^0 &= \varphi_n(u^0). \end{aligned}$$

We choose the normed spaces as $\mathcal{X}_n = \mathcal{Y}_n = \underbrace{\mathcal{K}_n \times \ldots \times \mathcal{K}_n}_{K+1}$, hence $\mathbf{v}_n := (\mathbf{v}^0, \ldots, \mathbf{v}^K) \in \mathcal{X}_n$. We define the mapping $F_n : \mathcal{X}_n \to \mathcal{Y}_n$ on any element $\mathbf{v}_n \in \mathcal{X}_n$ as

$$[F_n(\mathbf{v}_n)]_k = \begin{cases} \mathbf{v}^0 - \varphi_n(u^0), & k = 0, \\ \\ \mathbf{v}^k - Q_1^{-1} Q_2 \mathbf{v}^{k-1}, & k = 1, 2, \ldots, K \end{cases} \tag{2.16}$$

where $\mathbf{v}^0 := \mathbf{v}^0 - \varphi_n(u^0)$. From the relation (2.16) we can express $\mathbf{v}^k$ as

$$\mathbf{v}^k = \begin{cases} [F_n(\mathbf{v}_n)]_0 + \varphi_n(u^0), & k = 0, \\ \\ [F_n(\mathbf{v}_n)]_k + Q_1^{-1} Q_2 \mathbf{v}^{k-1}, & k = 1, 2, \ldots, K. \end{cases} \tag{2.17}$$

The first step to prove the N-stability property, we define the norm in $\mathcal{K}_n$ as

$$\left\| \mathbf{v}^k \right\|_{\mathcal{K}_n} = \left(\sum_{1 \le j \le n} |v^k(x_j)|^2 \right)^{1/2} = \left\| \mathbf{v}^k \right\|_2 .$$

In the sequel we calculate the exact value of $\left\| Q_1^{-1} Q_2 \right\|_2$. To this aim we proof the following lemma.

Lemma 2.4. *The following relation holds:*

$$\left\| Q_1^{-1} Q_2 \right\|_2 = 1. \tag{2.18}$$

Proof. The matrix D_p in (2.13) is skew-symmetric matrix ($D_p^* = -D_p$). Moreover, for an arbitrary matrix $M \in \mathbb{R}^{n\times n}$ we have the relation $\|M\|_2^2 = \rho(MM^*)$. Using these properties to (2.18), we obtain

$$\begin{aligned}
\left\|Q_1^{-1}Q_2\right\|_2^2 &= \left\|(I+D_p)^{-1}(I-D_p)\right\|_2^2 \\
&= \rho\Bigg((I+D_p)^{-1}(I-D_p)\Big[(I+D_p)^{-1}(I-D_p)\Big]^*\Bigg) \\
&= \rho\Bigg((I+D_p)^{-1}(I-D_p)(I-D_p)^*\Big[(I+D_p)^{-1}\Big]^*\Bigg) \\
&= \rho\Bigg((I+D_p)^{-1}(I-D_p)(I+D_p)\Big[(I+D_p)^{-1}\Big]^*\Bigg) \\
&= \rho\Bigg((I+D_p)^{-1}(I+D_p)(I-D_p)\Big[(I+D_p)^{-1}\Big]^*\Bigg) \\
&= \rho\Bigg((I-D_p)\Big[(I+D_p)^{-1}\Big]^*\Bigg) = \rho\Bigg((I+D_p)^{-1}(I-D_p)^*\Bigg) \\
&= \rho\Bigg((I+D_p)^{-1}(I+D_p)\Bigg) = 1.
\end{aligned}$$

This relation proves our statement. □

Before we start to prove the N-stability property, we give a useful norm relation which helps us how to choose properly the norms in $\mathcal{X}_n$ and $\mathcal{Y}_n$, respectively. Using (2.17) and Lemma 2.4, for $i = 0, \ldots, K$ we get

$$\begin{aligned}
\sum_{k=0}^{i} \left\|\mathbf{z}^k - \mathbf{w}^k\right\|_2 &\leq \sum_{k=0}^{i} \|[F_n(\mathbf{z}_n)]_k - [F_n(\mathbf{w}_n)]_k\|_2 + \sum_{k=0}^{i-1} \left\|\mathbf{z}^k - \mathbf{w}^k\right\|_2 \\
&\leq \sum_{k=0}^{K} \|[F_n(\mathbf{z}_n)]_k - [F_n(\mathbf{w}_n)]_k\|_2 + \sum_{k=0}^{i-1} \left\|\mathbf{z}^k - \mathbf{w}^k\right\|_2 \\
&\leq 2\sum_{k=0}^{K} \|[F_n(\mathbf{z}_n)]_k - [F_n(\mathbf{w}_n)]_k\|_2 + \sum_{k=0}^{i-2} \left\|\mathbf{z}^k - \mathbf{w}^k\right\|_2 \\
&\leq \ldots \\
&\leq K\sum_{k=0}^{K} \|[F_n(\mathbf{z}_n)]_k - [F_n(\mathbf{w}_n)]_k\|_2 + \left\|\mathbf{z}^0 - \mathbf{w}^0\right\|_2 \\
&= K\sum_{k=0}^{K} \|[F_n(\mathbf{z}_n)]_k - [F_n(\mathbf{w}_n)]_k\|_2 + \|[F_n(\mathbf{z}_n)]_0 - [F_n(\mathbf{w}_0)]_k\|_2 \\
&\leq (K+1)\sum_{k=0}^{K} \|[F_n(\mathbf{z}_n)]_k - [F_n(\mathbf{w}_n)]_k\|_2 \\
&\leq 2K\sum_{k=0}^{K} \|[F_n(\mathbf{z}_n)]_k - [F_n(\mathbf{w}_n)]_k\|_2 \,.
\end{aligned}$$

Hence, for $i = 0, \ldots, K$ we obtain the estimation

$$\sum_{k=0}^{i} \left\|\mathbf{z}^k - \mathbf{w}^k\right\|_2 \leq 2K\sum_{k=0}^{K} \|[F_n(\mathbf{z}_n)]_k - [F_n(\mathbf{w}_n)]_k\|_2 \,. \tag{2.19}$$

We introduce the following norms in $\mathcal{X}_n$ and $\mathcal{Y}_n$:

$$\begin{aligned}
&in\ \mathcal{X}_n : \|\mathbf{v}_n\|_{\mathcal{X}_n} = \delta \sum_{j=0}^{K} \left\|\mathbf{v}^j\right\|_{\mathcal{K}_n}, \\
&in\ \mathcal{Y}_n : \|\mathbf{v}_n\|_{\mathcal{Y}_n} = \sum_{j=0}^{K} \left\|\mathbf{v}^j\right\|_{\mathcal{K}_n}.
\end{aligned} \tag{2.20}$$

Then, based on (2.19) we get

$$\|\mathbf{z}_n - \mathbf{w}_n\|_{\mathcal{X}_n} \leq 2T \, \|F_n(\mathbf{z}_n) - F_n(\mathbf{w}_n)\|_{\mathcal{Y}_n} \, ,$$

where $\delta K = T$.

It is easy to see the above estimation is in the form of (2.4) with $S = 2T$, therefore we proved the validity of the following statement.

Theorem 2.5. *The centralized Crank–Nicolson-method is N-stable for the periodic initial-value transport problem* (2.6)-(2.8) *in the norm* (2.20)*.*

Remark 2.6. *In the norm* (2.20) *the order of the consistency of the centralized Crank–Nicolson-method is two both in time and space.*

Hence, using Theorems 2.2 and 2.5, we immediately get the following statement.

Corollary 2.7. *The centralized Crank–Nicolson-method is convergent for the periodic initial-value transport problem* (2.6)-(2.8) *and the order of the convergence is two both in time and space.*

Remark 2.8. *Based on estimation* (2.19) *we are able to define N-stability for other norms, too.*

- *The $S = 2T$ stability constant of* (2.4) *can reach if we define the norms as*
  - *in $\mathcal{X}_n$:* $\|\boldsymbol{v}_n\|_{\mathcal{X}_n} = \max\limits_{0 \leq j \leq K} \left\|\boldsymbol{v}^j\right\|_{\mathcal{K}_n}$,
  - *in $\mathcal{Y}_n$:* $\|\boldsymbol{v}_n\|_{\mathcal{Y}_n} = \sum\limits_{j=0}^{K} \left\|\boldsymbol{v}^j\right\|_{\mathcal{K}_n}$.
- *Using the relation*

  $$\sqrt{\sum_{j=0}^{K} \|\boldsymbol{v}^j\|^2_{\mathcal{K}_n}} \leq \sum_{j=0}^{K} \left\|\boldsymbol{v}^j\right\|_{\mathcal{K}_n}$$

  we can define the following norms:
  - *in $\mathcal{X}_n$:* $\|\boldsymbol{v}_n\|_{\mathcal{X}_n} = \delta \left(\sum\limits_{j=0}^{K} \left\|\boldsymbol{v}^j\right\|^2_{\mathcal{K}_n} \right)^{1/2}$,

– *in* $\mathcal{Y}_n$: $\|\boldsymbol{v}_n\|_{\mathcal{Y}_n} = \sum_{j=0}^{K} \|\boldsymbol{v}^j\|_{\mathcal{K}_n}$.

In this case the N-stability estimation (2.4) *is valid with* $S = 2T$.

Remark 2.9. *Consider the relation* (2.17)*. For the first term we can give a following estimation for* $i = 0, \ldots, K$

$$\|\boldsymbol{z}^i - \boldsymbol{w}^i\|_{\mathcal{K}_n} \le \|[F_n(\boldsymbol{z}_n)]_i - [F_n(\boldsymbol{w}_n)]_i\|_{\mathcal{K}_n} + \|\boldsymbol{z}^{i-1} - \boldsymbol{w}^{i-1}\|_{\mathcal{K}_n}.$$

Applying this iteration for $i = 0, \ldots, K$*, we gain the estimation*

$$\|\boldsymbol{z}^i - \boldsymbol{w}^i\|_{\mathcal{K}_n} \le \sum_{k=0}^{i} \|[F_n(\boldsymbol{z}_n)]_k - [F_n(\boldsymbol{w}_n)]_k\|_{\mathcal{K}_n} \le \sum_{k=0}^{K} \|[F_n(\boldsymbol{z}_n)]_k - [F_n(\boldsymbol{w}_n)]_k\|_{\mathcal{K}_n}.$$

Hence, we can define two more norms for which the N-stability property holds.

- *By choosing the norms as*
  - *in* $\mathcal{X}_n$: $\|\boldsymbol{v}_n\|_{\mathcal{X}_n} = \max_{0 \le j \le K} \|\boldsymbol{v}^j\|_{\mathcal{K}_n}$,
  - *in* $\mathcal{Y}_n$: $\|\boldsymbol{v}_n\|_{\mathcal{Y}_n} = \sum_{j=0}^{K} \|\boldsymbol{v}^j\|_{\mathcal{K}_n}$,

  and due to the obvious relation

  $$\max_{0 \le j \le K} \|\boldsymbol{v}^j\|_{\mathcal{K}_n} \le \sum_{j=0}^{K} \|[F_n(\boldsymbol{v}_n)]_j\|_{\mathcal{K}_n},$$

  we get the stability constant $S = 1$.

- *By defining the norms as*
  - *in* $\mathcal{X}_n$: $\|\boldsymbol{v}_n\|_{\mathcal{X}_n} = \delta \max_{0 \le j \le K} \|\boldsymbol{v}^j\|_{\mathcal{K}_n}$,
  - *in* $\mathcal{Y}_n$: $\|\boldsymbol{v}_n\|_{\mathcal{Y}_n} = \max_{0 \le j \le K} \|[F_n(\boldsymbol{v}_n)]_j\|_{\mathcal{K}_n}$,

  and since the relation

  $$\max_{0 \le j \le K} \|\boldsymbol{v}^j\|_{\mathcal{K}_n} \le K \max_{0 \le j \le K} \|[F_n(\boldsymbol{v}_n)]_j\|_{\mathcal{K}_n}$$

  holds, we have the N-stability with $S = 1$.

As we could see, the N-stability notion is useful from the application point of view. To prove this property to the reaction-diffusion (see more details in [3],[4]) and the transport problems, the key point is the proper definition of the φ_n and ψ_n mappings, the normed spaces of the discrete problems and the corresponding norms. It has been summarized in Table 2.2.

	Reaction-diffusion problem	Transport problem
$\varphi_n,\ \psi_n$	grid functions	grid functions
$\mathcal{K}_n$	VS of grid functions	VS of grid functions
$\mathcal{X}_n \equiv \mathcal{Y}_n$	$\underbrace{\mathcal{K}_n \times \ldots \times \mathcal{K}_n}_{K+1}$	$\underbrace{\mathcal{K}_n \times \ldots \times \mathcal{K}_n}_{K+1}$
$\left\Vert \mathbf{v}^k \right\Vert_{\mathcal{K}_n}$	$\max\limits_{1\le j\le n} \lvert v^k(x_j)\rvert$	$\left(\sum\limits_{1\le j\le n} \lvert v^k(x_j)\rvert^2\right)^{1/2}$
$\left\Vert \mathbf{v}_n \right\Vert_{\mathcal{X}_n}$	$\max\limits_{0\le k\le K} \left\Vert \mathbf{v}^k \right\Vert_{\mathcal{K}_n}$	$\delta \sum\limits_{j=0}^{K} \left\Vert \mathbf{v}^j \right\Vert_{\mathcal{K}_n}$
$\left\Vert \mathbf{v}_n \right\Vert_{\mathcal{Y}_n}$	$\left\Vert \mathbf{v}^0 \right\Vert_{\mathcal{K}_n} + \sum\limits_{j=1}^{K} \delta \left\Vert \mathbf{v}^j \right\Vert_{\mathcal{K}_n}$	$\sum\limits_{j=0}^{K} \left\Vert \mathbf{v}^j \right\Vert_{\mathcal{K}_n}$

Table 2.2: How to choose operators, normed spaces and corresponding norms to prove N-stability.

2.3. The N-stability of the transport problem with forcing term

Consider the periodic initial-value transport problem with forcing term, i.e.,

$$\partial_t u(t,x) + a\partial_x u(t,x) = f(t,x), \qquad x \in \mathbb{R},\ t \in [0,T], \tag{2.21}$$

$$u(t,x) = u(t,x+1), \quad x \in \mathbb{R},\ t \in [0,T] \tag{2.22}$$

$$u(0,x) = u^0(x), \qquad x \in \mathbb{R}, \tag{2.23}$$

where $T \in \mathbb{R}^+$ and $a \in \mathbb{R}$ are fixed constants and $f(t,x)$ is a given function. For this problem we get the equalities

$$\mathbf{u}^{k+1} - Q_1^{-1}Q_2\mathbf{u}^k = Q_1^{-1}\mathbf{f}^{k+1},\ k = 0,\ldots,K-1, \tag{2.24}$$

$$\mathbf{u}^0 = \varphi_n(u^0), \tag{2.25}$$

where $\mathbf{f}$ denotes the grid function defined on the grid points x_j for all $j = 1, \dots, n$. Introducing the element $\mathbf{g}_n \in \mathcal{Y}_n$ to right-hand sides we can define the F_n^g operator to the problem (2.21)-(2.23) as

$$F_n^g(\mathbf{v}_n) = F_n(\mathbf{v}_n) - \mathbf{g}_n.$$

We can easily proof the N-stability property. If the centralized Crank–Nicolson-method is stable, then for arbitrary $\mathbf{z}_n, \mathbf{w}_n \in \mathcal{X}_n$ the relation

$$\|\mathbf{z}_n - \mathbf{w}_n\|_{\mathcal{X}_n} \leq S \left\| F_n^g(\mathbf{z}_n) - F_n^g(\mathbf{w}_n) \right\|_{\mathcal{Y}_n}$$

holds. Since $F_n^g(\mathbf{v}_n) = F_n(\mathbf{v}_n) - \mathbf{g}_n$, we can rewrite the above estimation as

$$\|\mathbf{z}_n - \mathbf{w}_n\|_{\mathcal{X}_n} \leq S \left\| F_n(\mathbf{z}_n) - \mathbf{g}_n - F_n(\mathbf{w}_n) + \mathbf{g}_n \right\|_{\mathcal{Y}_n} = S \left\| F_n(\mathbf{z}_n) - F_n(\mathbf{w}_n) \right\|_{\mathcal{Y}_n}.$$

Then, according to Section 2.2, the stability relation holds with $S = 2T$. Thus, the periodic initial-value transport problem with forcing term is N-stable, too.

3. A further stability notion

In Theorem 2.2 we have shown that in case of consistency the N-stability is sufficient to guarantee the convergence. However, its necessity isn't clear. In this section we investigate this question. Using an example, taken from [9], we will show that the N-stability requirement is too restrictive.

3.1. Necessity of N-stability

Let $F_n^\alpha : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the operator given as

$$[F_n^\alpha(\mathbf{z})]_k = \begin{cases} \dfrac{z_k - z_{k-1}}{h} - z_{k-1}^2, & k = 1, 2, \dots, n \\ \\ z_0 - \alpha, & k = 0, \end{cases} \tag{3.26}$$

where h is the step-size parameter, $\alpha \in [0, 1)$ is some fixed constant and $nh = 1$. Taking the function $\bar{z}^\alpha(t) = \alpha/[1 - \alpha t]$, where $t \in [0, 1]$ and applying the φ_n as a grid function to the function $\bar{z}^\alpha(t)$, we get

$$[\varphi_n(\bar{z}^\alpha)]_k \equiv (\bar{\mathbf{z}}_n^\alpha)_k \equiv \bar{z}^\alpha(t_k) \equiv \frac{\alpha}{1 - \alpha t_k}, \qquad k = 0, 1, \dots, n,$$

where t_k are the grid points.

Remark 3.1. *With the discrete operator (3.26) the problem* $F_n^\alpha(u_n) = 0$ *can be considered as the discretization of the prototype of the simple Ricatti equation:*

$$\begin{cases} u'(t) = u^2(t), \quad t \in [0,1] \\ \\ u(0) = \alpha, \end{cases} \tag{3.27}$$

by mean of the forward Euler's rule on the equidistant mesh. Clearly, the solution of the problem (3.27) is the function $\bar{z}^\alpha$.

Substituting $\bar{\mathbf{z}}_n^\alpha$ into (3.26), we gain

$$[F_n^\alpha(\bar{\mathbf{z}}_n^\alpha)]_k = \begin{cases} \dfrac{\bar{z}^\alpha(t_k) - \bar{z}^\alpha(t_{k-1})}{h} - [\bar{z}^\alpha(t_{k-1})]^2, & k = 1, 2, \ldots, n \\ \\ (\bar{z}_n^\alpha)_0 - \alpha, & k = 0. \end{cases}$$

Let $\bar{\mathbf{w}}_n \in \mathbb{R}^{n+1}$ be a vector with the components w_k, such that $[F_n(\bar{\mathbf{w}}_n)] = 0$, where

$$[F_n(\bar{\mathbf{w}}_n)]_k = \begin{cases} \dfrac{w_k - w_{k-1}}{h} - w_{k-1}^2, & k = 1, 2, \ldots, n \\ \\ w_0 - 1, & k = 0. \end{cases}$$

We introduce the norms

$$\|\mathbf{x}_k\|_{\mathcal{X}_n} = \max_{1 \le k \le n+1} |x_k|,$$

$$\|\mathbf{y}_k\|_{\mathcal{Y}_n} = |y_0| + \sum_{k=1}^{n} h|y_k|,$$

respectively. We prove that (2.4) cannot be true for any stability constant S, which is independent of the mesh size. To to this aim, we show that the estimation

$$\|\bar{\mathbf{z}}_n^\alpha - \bar{\mathbf{w}}_n\|_{\mathcal{X}_n} \le S \, \|F_n^\alpha(\bar{\mathbf{z}}_n^\alpha) - F_n(\bar{\mathbf{w}}_n)\|_{\mathcal{Y}_n} \tag{3.28}$$

cannot be hold uniformly for all n. Since $(\bar{w}_n)$ is defined by the recursion $\bar{\mathbf{w}}_n = \bar{\mathbf{w}}_{n-1} + h\bar{\mathbf{w}}_n^2$, due to [13], the approximation at the last grid point $t = 1$ behaves like $1/(h|\ln h|)$. Thus,

$$\lim_{n\to\infty} (\bar{\mathbf{w}}_n)_n = \lim_{h\to 0} \frac{1}{h|\ln h|} = \infty.$$

Since $(\bar{\mathbf{z}}_n^\alpha)_n \equiv \alpha/[1-\alpha]$ and $\alpha \in [0,1)$, hence the value of $(\bar{\mathbf{z}}_n^\alpha)_n$ is finite. So the left term of (3.28) converges to ∞ as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \|\bar{\mathbf{z}}_n^\alpha - \bar{\mathbf{w}}_n\|_{\mathcal{X}_n} = \infty. \tag{3.29}$$

For the right-hand side of (3.28) we have

$$[F_n^\alpha(\bar{\mathbf{z}}_n^\alpha) - \underbrace{F_n(\bar{\mathbf{w}}_n)}_{=0}]_k = \begin{cases} \dfrac{\bar{z}^\alpha(t_k) - \bar{z}^\alpha(t_{k-1})}{h} - [\bar{z}^\alpha(t_{k-1})]^2, & k = 1, 2, \ldots, n \\ \alpha - 1, & k = 0. \end{cases} \tag{3.30}$$

This means that in the normed space $\mathcal{Y}_n$ we have to define the local discretization error. The idea of the proof is based on the work [17].

Lemma 3.2. *Let consider the Cauchy problem*

$$u'(t) = f(u(t)) \tag{3.31}$$
$$u(0) = u_0, \tag{3.32}$$

where $t \in [0,1]$, $u_0 \in \mathbb{R}$ and $f \in C^1(\mathbb{R})$. Then for the problem (3.31)-(3.32) the local discretization error of the forward Euler method on equidistant mesh can be estimated as

$$l_n(\bar{u})(t_i) \leq \frac{M_2(\bar{u})}{2} h,$$

where $t_i = ih$ $(i = 1, \ldots, n)$, $M_2(\bar{u}) := \sup_{t\in(0,1)} |\bar{u}''(t)| < \infty$ and h is the step-size of the mesh.

Proof. We have the relation

$$\begin{aligned} l_n(\bar{u})(t_i) &= [F_n(\varphi_n(\bar{u}))](t_i) = \frac{\bar{u}(t_i) - \bar{u}(t_{i-1})}{h} - \bar{u}'(t_{i-1}) \\ &\leq \max_{1\leq i\leq n} \Big|\bar{u}'((i-1)h) - \frac{1}{h}\Big(\bar{u}(ih) - \bar{u}((i-1)h)\Big)\Big| \\ &= \max_{1\leq i\leq n} \Big|\frac{1}{h}\int_{(i-1)h}^{ih} \bar{u}'((i-1)h) - \bar{u}'(s)\mathrm{ds}\Big| \\ &\leq \frac{1}{h}\max_{1\leq i\leq n} \int_{t_{i-1}}^{t_i} |\bar{u}'(t_{i-1}) - \bar{u}'(s)|\mathrm{ds}. \end{aligned}$$

Hence,

$$l_n(\bar{u})(t_i) \leq \frac{1}{h} M_2(\bar{u}) \frac{1}{2} h^2 = \frac{M_2(\bar{u})}{2} h.$$

$\square$

Using the introduced norm in $\mathcal{Y}_n$ to (3.30) and Lemma 3.2, we get

$$\|F_n^\alpha(\bar{\mathbf{z}}_n^\alpha) - F_n(\bar{\mathbf{w}}_n)\|_{\mathcal{Y}_n} = |\alpha - 1| + \sum_{k=1}^{n} h \cdot l_n(\bar{z}^\alpha(t_k)) \leq |\alpha - 1| + \frac{M_2(\bar{z}^\alpha)}{2}.$$

Thus,

$$\lim_{n\to\infty} \|F_n^\alpha(\bar{\mathbf{z}}_n^\alpha) - F_n(\bar{\mathbf{w}}_n)\|_{\mathcal{Y}_n} < \infty. \tag{3.33}$$

From (3.29) and (3.33) we can see the estimation (3.28) cannot hold. This means that the discretization is not N-stable.

Thus, the statement of Theorem 2.2 cannot be satisfied. However, we will see through the numerical results that the forward Euler method on the equidistant mesh will converge to the solution of the problem (3.27). To demonstrate this, we select the value $\alpha = 0.8$ in (3.27), and we apply the forward Euler method to this problem. The results have been summarized in Figure 3.2 and Table 3.3. The obtained numerical results suggest the convergence of the method.

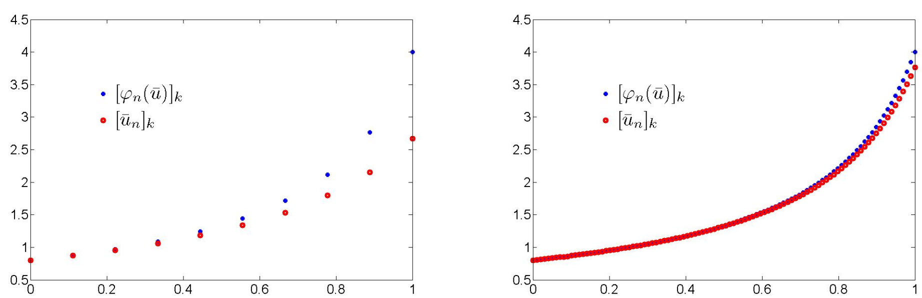

Figure 3.2: The restricted true solution and the numerical solution for 10 and 100 grid points to the problem (3.27).

Number of grid points	$\|e_n\|_{\mathcal{X}_n}$
10^1	$1.5175 \cdot 10^1$
10^2	$5.8687 \cdot 10^{-1}$
10^4	$6.0863 \cdot 10^{-2}$
10^6	$6.0887 \cdot 10^{-3}$

Table 3.3: The global discretization error in the introduced norm to the problem (3.27).

3.2. K-stability and its application

This example shows that the N-stability definition is too restrictive, because we require the condition (2.4) for any elements from $\mathcal{D}_n$. It also shows that if $\bar{\mathbf{w}}_n$ is far from $\bar{\mathbf{z}}_n^\alpha$ (i.e., the perturbation $\bar{\mathbf{z}}_n^\alpha$ is too large), then the estimate (2.4) cannot hold.

This motivates to introduce the idea of local stability and stability threshold notions [7].

Definition 3.3. *The discretization $\mathscr{D}$ is called* K-stable *for the problem $\mathscr{P}$ at the element $\bar{u} \in \mathcal{X}$ if there exist constants $S \in \mathbb{R}$ and $\mathcal{R} \in (0, \infty]$ such that*

- *$B_{\mathcal{R}}(\varphi_n(\bar{u})) \subset \mathcal{D}_n$ holds from some index,*
- *for all z_n, $w_n \in B_{\mathcal{R}}(\varphi_n(\bar{u}))$ the estimate*

$$\|z_n - w_n\|_{\mathcal{X}_n} \leq S \, \|F_n(z_n) - F_n(w_n)\|_{\mathcal{Y}_n} \tag{3.34}$$

holds.

We summarize the main theoretical result of the K-stability notion, based on the work of [1].

Theorem 3.4. *We assume that*

- *the discretization $\mathscr{D}$ is consistent and K-stable at $\bar{u}$ with stability threshold $\mathcal{R}$ and constant S on problem $\mathscr{P}$,*
- *the numerical method possesses the property* $\dim \mathcal{X}_n = \dim \mathcal{Y}_n < \infty$*,*

- *F_n is continuous on the ball $B_R(\varphi_n(\bar{u}))$.*

Then

(a) the discretization $\mathscr{D}$ generates a numerical method such that equation (1.2) has a unique solution in $B_{\mathcal{R}}(\varphi_n(\bar{u}))$ from some index,

(b) the discretization $\mathscr{D}$ is convergent on problem $\mathscr{P}$ and the order of convergence is not less than the order of consistency.

Proof. The proofs has been given in Lemma 25 and Theorem 26 in [1]. $\square$

Remark 3.5. *Theorem 3.4 guarantees that equation* (1.2) *has a unique solution in some suitably chosen ball. This means that the K-stability in the nonlinear case locally satisfies those properties what the linear stability notion (or, equivalently, the N-stability notion for the linear case) does.*

In the sequel we examine the K-stability for a general class of operators.

Let $F_n : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the operator given as

$$[F_n(\mathbf{z})]_k = \begin{cases} \dfrac{z_k - z_{k-1}}{h} - f(z_{k-1}), & k = 1, 2, \ldots, n \\ \\ z_0 - u_0, & k = 0, \end{cases} \tag{3.35}$$

where h is the step-size parameter, $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function and u_0 is some fixed value. The discretization (3.35) is the application of the forward Euler method on the equidistant mesh to the autonomous Cauchy problem

$$\begin{cases} u'(t) = f(u(t)), & t \in [0, 1] \\ \\ u(0) = u_0. \end{cases} \tag{3.36}$$

Let $\mathcal{R} > 0$ and $B_{\mathcal{R}} = \cup_{t\in[0,1]}[u(t) - \mathcal{R}, u(t) + \mathcal{R}]$. The function f is Lipschitz continuous on $B_{\mathcal{R}}$ with constant $L(\mathcal{R})$. We consider only those vectors $\mathbf{z}_n$, $\mathbf{w}_n$ for which

$$\|\mathbf{z}_n - \varphi_n(\bar{u})\|_{\mathcal{X}_n} \leq \mathcal{R}$$

and

$$\|\mathbf{w}_n - \varphi_n(\bar{u})\|_{\mathcal{X}_n} \leq \mathcal{R}.$$

These conditions implies that $(\mathbf{z}_n)_k$, $(\mathbf{w}_n)_k \in B_{\mathcal{R}}$, where the Lipschitz condition holds. Then we substitute $\mathbf{z}_n$ and $\mathbf{w}_n$ into (3.35). The subtraction of $[F_n(\mathbf{z}_n)]_k$ and $[F_n(\mathbf{w}_n)]_k$ leads to the equality

$$\begin{aligned}(\mathbf{z}_n)_k - (\mathbf{w}_n)_k = (\mathbf{z}_n)_{k-1} - (\mathbf{w}_n)_{k-1} + h\Big([f(\mathbf{z}_n)]_{k-1} - [f(\mathbf{w}_n)]_{k-1}\Big)\\ + h\Big([F_n(\mathbf{z}_n)]_k - [F_n(\mathbf{w}_n)]_k\Big).\end{aligned}$$

Using the Lipschitz condition we gain

$$|(\mathbf{z}_n)_k - (\mathbf{w}_n)_k| \leq (1 + hL(\mathcal{R}))|(\mathbf{z}_n)_{k-1} - (\mathbf{w}_n)_{k-1}| + h|[F_n(\mathbf{z}_n)]_k - [F_n(\mathbf{w}_n)]_k|.$$

Then, by induction we get

$$\|\mathbf{z}_n - \mathbf{w}_n\|_{\mathcal{X}_n} = \max_{0\leq k\leq n} |(\mathbf{z}_n)_k - (\mathbf{w}_n)_k| \leq e^{L(\mathcal{R})}\|F_n(\mathbf{z}_n) - F_n(\mathbf{w}_n)\|_{\mathcal{Y}_n}. \quad (3.37)$$

The estimation (3.37) is in the form of (3.34), i.e., the discretization - which is consistent - is K-stable with constant $S = e^{L(\mathcal{R})}$.

Theorem 3.6. *The discrete operator* (3.35) *under the given conditions is K-stable.*

Hence, in virtue of Theorems 3.4 and 3.6, the following statement is true.

Corollary 3.7. *The sequence of the solutions of the problems $F_n(\boldsymbol{z}_n) = 0$, where F_n is defined by (3.35), is convergent to the solution of the Cauchy problem* (3.36).

Remark 3.8. *We recall the discretization (3.26) and the problem (3.27). As we have seen in Section 3.1, the discretization isn't N-stable. However, if we choose $f(u(t)) \equiv u^2(t)$ and $u_0 \equiv \alpha \in [0, 1)$ in Theorem 3.6, it is easy to see that the discretization is K-stable.*

Remark 3.9. *Let $\mathcal{R} > 0$ fixed. Then, as we have seen in Section 3.1, the condition $\bar{\boldsymbol{v}}_n^\alpha, \bar{\boldsymbol{w}}_n \in B_{\mathcal{R}}(\bar{\boldsymbol{v}}_n^\alpha)$ cannot be guaranteed. However, if we require the stability condition only for the elements from $B_{\mathcal{R}}(\bar{\boldsymbol{v}}_n^\alpha)$ (that is the stability notion in Definition 3.3 as we have seen in the previous example for a general class of operators), then the condition* (3.34) *is satisfied.*

In a similar way we examine the K-stability for a more general class of discrete operators. Let $F_n^\theta : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the operator given as

$$[F_n^\theta(\mathbf{z})]_k = \begin{cases} \dfrac{z_k - z_{k-1}}{h} - (1-\theta) f(z_{k-1}) - \theta f(z_k), & k = 1, 2, \ldots, n \\ \\ z_0 - u_0, & k = 0, \end{cases} \tag{3.38}$$

where $\theta \in [0,1]$ is given parameter, h denotes the step-size, $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function and u_0 is some fixed value. The discretization (3.35) can be ed as the application of the standard θ-method on the equidistant mesh to the problem (3.36).

In the previous train of thought we get the equality

$$\begin{aligned}(\mathbf{z}_n)_k - (\mathbf{w}_n)_k = (\mathbf{z}_n)_{k-1} - (\mathbf{w}_n)_{k-1} + h(1-\theta)\Big([f(\mathbf{z}_n)]_{k-1} - [f(\mathbf{w}_n)]_{k-1}\Big) \\ + h\theta\Big([f(\mathbf{z}_n)]_k - [f(\mathbf{w}_n)]_k\Big) + h\Big([F_n^\theta(\mathbf{z}_n)]_k - [F_n^\theta(\mathbf{w}_n)]_k\Big).\end{aligned}$$

Using the Lipschitz condition we obtain

$$\begin{aligned}\Big|(\mathbf{z}_n)_k - (\mathbf{w}_n)_k\Big| \le \frac{1 + h(1-\theta)L(\mathcal{R})}{1 - h\theta L(\mathcal{R})}\Big|(\mathbf{z}_n)_{k-1} - (\mathbf{w}_n)_{k-1}\Big| \\ + \frac{1}{1 - h\theta L(\mathcal{R})} h \Big|[F_n^\theta(\mathbf{z}_n)]_k - [F_n^\theta(\mathbf{w}_n)]_k\Big|.\end{aligned}$$

Hereinafter, based on [2], we give an estimation for $(1 - h\theta L(\mathcal{R}))^{-1}$. For the values h, satisfying the condition $h\theta L(\mathcal{R}) \in [0, 0.5]$, we have

$$1 \le \frac{1}{1 - h\theta L(\mathcal{R})} = 1 + h\theta L(\mathcal{R}) + (h\theta L(\mathcal{R}))^2 \frac{1}{1 - h\theta L(\mathcal{R})}.$$

Hence, the estimation

$$\frac{(h\theta L(\mathcal{R}))^2}{1 - h\theta L(\mathcal{R})} \le h\theta L(\mathcal{R})$$

holds. Therefore, we have the upper bound

$$\frac{1}{1 - h\theta L(\mathcal{R})} \le 1 + 2h\theta L(\mathcal{R}).$$

Thus, we can give the following estimation:

$$\Big|(\mathbf{z}_n)_k - (\mathbf{w}_n)_k\Big| \le (1 + 2h\theta L(\mathcal{R}))\Big[(1 + h(1-\theta)L(\mathcal{R}))\Big|(\mathbf{z}_n)_{k-1} - (\mathbf{w}_n)_{k-1}\Big| \\ + h\Big|[F_n^\theta(\mathbf{z}_n)]_k - [F_n^\theta(\mathbf{w}_n)]_k\Big|\Big].$$

Then, by induction we get

$$\|\mathbf{z}_n - \mathbf{w}_n\|_{\mathcal{X}_n} = \max_{0\le k\le n} |(\mathbf{z}_n)_k - (\mathbf{w}_n)_k| \le e^{(1+\theta)L(\mathcal{R})}\|F_n^\theta(\mathbf{z}_n) - F_n^\theta(\mathbf{w}_n)\|_{\mathcal{Y}_n}. \tag{3.39}$$

The estimation (3.39) proves the validity of the following statement.

Theorem 3.10. *The discrete operator* (3.38) *is K-stable with the stability constant* $S = e^{(1+\theta)L(\mathcal{R})}$.

Due to the consistency, in virtue of Theorems 3.4 and 3.10, the following statement is true.

Corollary 3.11. *The sequence of the solutions of the problems* $F_n^\theta(\boldsymbol{z}_n) = 0$*, where* F_n^θ *is defined by (3.38), is convergent to the solution of the Cauchy problem* (3.36).

4. Summary

In this paper our primary aim was to give and analyze the N- and K-stability concepts. The connection between the N-stability concept and the linear stability notion was investigated. We have shown that this approach provides the basic theorem of the numerical analysis, i.e., in case of consistency the convergence is guaranteed. At the same time, by giving an example, we have shown the insufficiency of the N-stability notion to the convergence. This fact motivated us to introduce a further stability concept, the K-stability notion. This notion has a local character and it is a natural extension of the N-stability.

These alternative stability concepts have several advantageous properties. First of all, both stability concepts ensure the basic theorem of the numerical analysis on the convergence. A further important property of this approach is very practical. Namely, using these concepts, we are able to offer a new and effective alternative tool (in contrast with the well-known discrete time Fourier transform technique, see more details in [16]) for verifying the stability, and hence, convergence property for time-dependent problems. Particularly, in this paper the convergence of the centralized Crank–Nicolson-method to the transport problem were given in a compact form. Comparing the N- and K-stability concepts, we mention an important theoretical advantage of the K-stability. Namely, the K-stability (together with consistency) guarantees not only the convergence, but also the existence of the unique solution of the discrete problem (1.2) in a convenient ball (if the ball contains $\mathcal{D}_n$, then this result valid for N-stability, too).

References

[1] I. Faragó, M. E. Mincsovics, I. Fekete, Notes on the basic notions in nonlinear numerical analysis, EJQTDE, Proc. 9'th Coll. QTDE (2012), No. 6, pp. 1–22.

[2] I. Faragó, Convergence and stability constant of the theta-method, Application of Mathematics 2013, Edited by: J. Brandts, S. Korotov, M. Krzek, J. Sstek, T. Vejchodsky, pp. 42–51.

[3] I. Fekete, I. Faragó, A stability approach for reaction-diffusion problems, Applied Computational Intelligence and Informatics (SACI), 8th IEEE International Symposium (2013), pp. 191–195.

[4] I. Fekete, I. Faragó, N-stability of the θ-method for reaction-diffusion problems (submitted)

[5] M. K. Gavurin, Lectures on methods of computing (in Russian), Nauka, Moscow, 1971.

[6] L. V. Kantorovich, Functional analysis and applied mathematics (in Russian), UMN3, 28 (1948), No. 6, pp. 89–185.

[7] H. B. Keller, Approximation Methods for Nonlinear Problems with Application to Two-Point Boundary Value Problems, Math. Comput., 130 (1975), pp. 464–474.

[8] P. D. Lax, R. D. Richtmyer, Survey of the stability of linear finite difference equations, Comm. Pure Appl. Math., 9 (1956), pp. 267–293.

[9] J. C. López-Marcos, J. M. Sanz-Serna, A definition of stability for nonlinear problems, Numerical Treatment of Differential Equations, 104 (1988), pp. 216–226.

[10] T. Ortega, J. M. Sanz-Serna, Nonlinear stability and convergence of finite-difference methods for the "good" Boussinesq equation, Numer. Math. 58 (1990), No. 2, pp. 215-229.

[11] A. A. Samarskii, P. P. Matus, P. N. Vabishchevich, Differences schemes with operator factors, Mathematics and its Applications, 546, Kluwer Academic Publishers, Dordrecht, 2002.

[12] J. M. Sanz-Serna, C. Palencia, A general equivalence theorem in the theory of discretization methods, Math. Comp., 45 (1985), No. 171, pp. 143–152.

[13] J. M. Sanz-Serna, J. G. Verwer, A Study of the Recursion $y_{n+1} = y_n + \tau y_n^m$, J. Math. Anal. Appl., 116 (1986), pp. 456–463.

[14] J. M. Sanz-Serna, Two topics in nonlinear stability. Advances in numerical analysis, Oxford Sci. Publ., Oxford Univ. Press, New York, 1991, Vol. I, pp. 147–174.

[15] H. J. Stetter, Analysis of Discretization Methods for Ordinary Differential Equations, Springer, Berlin, 1973.

[16] J. W. Thomas, Numerical partial differential equations: finite difference methods, Texts in Applied Mathematics, 22, 2nd printing, Springer-Verlag, New York, 1998.

[17] V. A Trenogin, Functional Analysis (in Russian), Nauka, Moscow, 1980.